An HJM approach for multiple yield curves

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Multi-curve setting

- The underlying of basis interest rate instruments, such as
  - forward rate agreements,
  - swaps,
  - caplets,

  are Euribor or Libor rates for some (future) interval $[T, T+\delta]$, where the tenor δ is typically 1D, 1M, 3M, 6M, 1Y etc.

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- Before the financial crisis these yield curves coincided (more or less), but nowadays they differ significantly due to credit and liquidity risk of the interbank sector.
- In particular, the Euribor cannot be considered risk-free any longer.

$\Rightarrow$ Term structure models for multiple yield curves are required.

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  - ...to provide (semi-)analytic pricing formulas for interest rate derivatives
- Analysis of the relation to other multi-yield curve models

Eonia rate and overnight index swap (OIS) rates

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- OIS rates are the market quotes for these swaps. They are available for maturities ranging from 1 week to 60 years.

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- OIS rates are assumed to be the best proxy for riskfree rates and constitute a bootstrapping instruments to obtain (at time t) the curve of
  - riskfree bond prices : $T \mapsto B(t, T)$.
  - riskfree forward rates: $T \mapsto f_t(T) = -\partial_T \log B(t, T)$.
  - OIS-FRA rates for $[T, T + \delta]$

$$T \mapsto L_t^D(T, T + \delta) = \frac{1}{\delta} \left(\frac{B(t, T)}{B(t, T + \delta)} - 1 \right).$$

Note that $L_t^D(T, T + \delta)$ is the analog of the pre-crisis (riskfree simply compounded) forward Euribor rate for $[T, T + \delta]$.

Euribor rates and FRA rates

- $L_T(T, T+\delta)$: Euribor rate at time T with maturity $T+\delta$:
  - rate at which Euro interbank term deposits of length δ are being offered by one prime bank to another,
  - trimmed average rates submitted by panel of banks for 15 maturities with corresponding tenor $\delta \in \{1/52, 2/52, 3/52, 1/12, 2/12, \ldots, 1\}$.

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- $L_t(T, T+\delta)$: FRA rate at time t for $[T, T+\delta]$:
  - rate K fixed at time t such that the value of the FRA contract, whose payoff at time $T+\delta$ is $L_T(T, T+\delta) - K$, has value 0:

$$L_t(T, T+\delta) = \mathbb{E}_{\mathbb{Q}^{T+\delta}}\left[L_T(T, T+\delta) | \mathcal{F}_t\right],$$

    where $\mathbb{Q}^{T+\delta}$ denotes the $T+\delta$ forward measure with numeraire $B(t, T+\delta)$.
  - For each tenor δ, the term structure of the FRA rates $T \mapsto L_t(T, T+\delta)$ is constructed from the market interest rate instruments (swaps, etc.) linked to the Euribor with the corresponding tenor.

FRA rates in the multi-curve setting

- In the multi-curve setting, FRA rates are typically higher than riskfree OIS-FRA rates:

$$\begin{aligned} L_t(T, T+\delta) > L_t^D(T, T+\delta) &= \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right) \\ &= \frac{1}{\delta}\left(\prod_{i=1}^{360\delta}(1 + \frac{1}{360}L_t(T_i, T_i + 1/360)) - 1\right), \end{aligned}$$

where $L_t(T_i, T_i + 1/360)$ denotes the Eonia FRA rate.

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$$= \frac{1}{\delta}\left(\prod_{i=1}^{360\delta}\left(1 + \frac{1}{360} L_t(T_i, T_i + 1/360)\right) - 1\right),$$

  where $L_t(T_i, T_i + 1/360)$ denotes the Eonia FRA rate.

- This is related to the fact that the composition of the EURIBOR panel is updated over time to include only creditworthy banks.
  - The rates obtained from OIS reflect the average credit quality of a periodically refreshed pool of creditworthy banks.
  - EURIBOR rates incorporate the risk that the average credit quality of an initial set of creditworthy banks will deteriorate over the term of the loan.

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- For risky curves, the bootstrapping standard and closest to market data are the FRA curves

$$T \mapsto L_t(T, T + \delta)$$

(one curve for each tenor δ).

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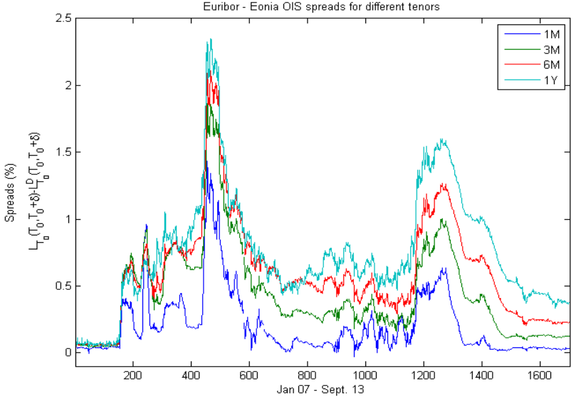
  - Multiplicative spread between riskfree and risky forward prices:
$$S^\delta(t, T) := \frac{1 + \delta L_t(T, T+\delta)}{1 + \delta L_t^D(T, T+\delta)} = \frac{B^\delta(t, T)B(t, T+\delta)}{B^\delta(t, T+\delta)B(t, T)}$$
where the (artificial) risky bond prices are defined via $\frac{B^\delta(t,T)}{B^\delta(t,T+\delta)} = (1 + \delta L_t(T, T+\delta))$.

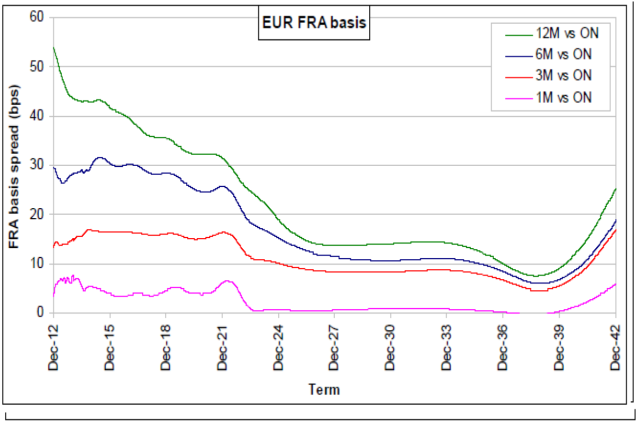
OIS Eonia-Euribor spreads

- Additive OIS Eonia - Euribor spread $L_T(T, T+\delta) - L_T^D(T, T+\delta)$ from Jan. 2007 to September 2013 for $\delta = 1/12, 3/12, 6/12, 1$:



Term structure of FRA spreads

- Spreads of OIS-FRA rates vs. FRA rates $L_{T_0}(T, T+\delta) - L^D_{T_0}(T, T+\delta)$ at $T_0 = 11.12.12$ for $\delta = 1/12, 3/12, 6/12, 1$:



Literature

- Post-crisis interest rate market: Moreni and Pallavicini, 2010, Henrard 2007, Fujii et al. 2010, Chibane and Sheldon 2009, Ametrano and Bianchetti 2009, etc.
- Short rate approach: Kijima et al. 2009, Kenyon 2010, Filipović and Trolle 2012, etc.
- LIBOR Market model approach: Mercurio 2010, Grbac et al. 2013, etc.
- HJM approach: Moreni and Pallavicini 2010, Pallavini and Tarenghi, 2010, Fujii et al. 2009, Crepey et al. 2013, Chiarella et al. 2010, etc.

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- The curves which are the easiest to obtain from market data are $T \mapsto L_t(T, T+\delta)$ (short maturities are directly quoted). The classical HJM setting provides a term structure model for $T \mapsto f_t(T) \approx L_t(T, T+\frac{1}{360})$. $\Rightarrow$ Model for $T \mapsto L_t(T, T+\delta)$ is required.

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- Possibility to model either $L_t(T, T+\delta)$ or certain spreads between $L_t(T, T+\delta)$ and $L_t^D(T, T+\delta)$. $\Rightarrow$ Model for spreads to guarantee $L_t(T, T+\delta) \geq L_t^D(T, T+\delta)$ and an ordering of the spreads for different δ if desired.

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$\Rightarrow$ Model $T \mapsto S^\delta(t, T)$ for every tenor δ together with the classical HJM model for $T \mapsto f_t(T)$.

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- Supposing differentiability of $T \mapsto \log(S(t,T))$ a.s., we can represent $S(t,T)$ by

$$S(t,T) = e^{Z_t + \int_t^T \eta_t(s)ds},$$

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- We call Z the log-spot and $\eta_t(T)$ generalized forward rate.
- Advantage: Modeling is split into modeling the spot quantity and a generalized forward rate.

HJM-type models

Definition (cf. Kallsen -Krühner 2013, for option surface models)

A quintuple $(Z, \eta_0, \alpha, \sigma, X)$ is called HJM-type model for a family of positive semimartingales $\{(S(t,T))_{t\in[0,T]}, T \geq 0\}$ if

1. (X, Z) is a $d+1$-dimensional Itô-semimartingale (absolutely continuous characteristics)
2. $\eta_0 \colon \mathbb{R}_+ \to \mathbb{R}$ is measurable and $\int_0^T |\eta_0(t)|dt < \infty$ for any $T \in \mathbb{R}_+$,
3. $(\omega, t, T) \mapsto \alpha_t(T)(\omega)$ and $(\omega, t, T) \mapsto \sigma_t(T)(\omega)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+)$ measurable $\mathbb{R}$-and $\mathbb{R}^d$-valued processes and satisfy certain integrability conditions,
4. the generalized forward rate $\eta_t(T)$ has a regular decomposition given by
$$\eta_t(T) = \eta_0(T) + \int_0^t \alpha_s(T)ds + \int_0^t \sigma_s(T)dX_s,$$
5. $\{(S(t,T))_{t\in[0,T]}, T \geq 0\}$ satisfies $S(t,T) = e^{Z_t + \int_t^T \eta_t(s)ds}$, in particular $S(t,t) = e^{Z_t}$.

Remark on HJM type models

- $(S(t,T))_{t\in[0,T]}$ often corresponds to the evolution of the price of a derivative with maturity T and is thus a (local) martingale under some equivalent measure.
- The martingale property of $S(t,T)_{t\in[0,T]}$ can be characterized in terms of a drift condition on α and a consistency condition. For this we need the notion of the local exponent of a semimartingale.

Local exponents of semimartingales

Definition

Let X be an $\mathbb{R}^d$-valued semimartingale and β an $\mathbb{R}^d$-valued predictable X-integrable process. A predictable $\mathbb{R}$-valued process $(\Psi_t^X(\beta))_t$ is called local exponent of X at β (or Laplace cumulant process) if

$$\left(\exp\left(\int_0^t \beta_s dX_s - \int_0^t \Psi_s^X(\beta) ds\right)\right)_t$$

is a local martingale. We denote by $\mathcal{U}^X$ the set of processes β such that the local exponent exists.

Local exponents of semimartingales

Proposition

Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics (b, c, K). Let β be an $\mathbb{R}^d$-valued predictable X-integrable process. Then there is an equivalence between

- $\beta \in \mathcal{U}^X$,
- $\int_0^\cdot \beta_s dX_s$ *is an exponentially special semimartingale, that is* $e^{\int_0^\cdot \beta_s dX_s}$ *is a special semimartingale,*
- $\int_0^t \int_{\beta_s^\top \xi > 1} e^{\beta_s^\top \xi} K_s(d\xi) ds < \infty$ *a.s for all* $t > 0$.

In this case

$$\Psi_t^X(\beta_t) = \beta_t^\top b_t + \frac{1}{2}\beta_t^\top c_t \beta_t + \int (e^{\beta_t^\top \xi} - 1 - \beta_t^\top \chi(\xi)) K_t(d\xi).$$

HJM framework - drift and consistency condition

Theorem (cf. Kallsen and Krühner 2013)

For an HJM-type model the following conditions are equivalent:

1. $(S(t,T))_t$ *are martingales for all* $T \geq 0$.
2. *The so-called conditional expectation hypothesis holds:*

$$\mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right] = e^{Z_t + \int_t^T \eta_t(s)ds}$$

3. *The following conditions are satisfied:*
   - *martingale property of* $\left(\exp\left(Z_t + \int_0^t \left(\int_s^T \sigma_s(u)du\right) dX_s - \int_0^t \Psi_s^{Z,X}\left(1, \int_s^T \sigma_s(u)du\right) ds\right)\right)_{t\in[0,T]}$
   - *consistency condition:* $\Psi_t^Z(1) = \eta_{t-}(t)$,
   - *HJM drift condition:* $\int_t^T \alpha_t(s)ds = \Psi_t^Z(1) - \Psi_t^{Z,X}\left(1, \int_t^T \sigma_t(s)ds\right)$.

HJM framework - Remarks

- $(S(t,T))_t$ are local martingales if and only if the drift and the consistency condition is satisfied together with the local the martingale property of

$$\left(\exp\left(Z_t + \int_0^t \left(\int_s^T \sigma_s(u)du\right) dX_s - \int_0^t \Psi_s^{Z,X}\left(1, \int_s^T \sigma_s(u)du\right) ds\right)\right)_{t\in[0,T]}. \tag{1}$$

  The latter condition is equivalent to $Z_t + \int_0^t \left(\int_s^T \sigma_s(u)du\right) dX_s$ being an exponentially special semimartingale.

- A sufficient condition for (1) being a true martingale is

$$\sup_{t\leq T}\mathbb{E}\Bigg[\exp\left(\frac{1}{2}\left(1, \int_t^T \sigma_t^\top(u)du\right) c_t^{Z,X}\left(1, \int_t^T \sigma_t^\top(u)du\right)^\top\right)$$
$$\times \exp\left(\int \left(e^{\left(1,\int_t^T \sigma_t^\top(u)du\right)\xi}\left(1 - \left(1, \int_t^T \sigma_t^\top(u)du\right)\xi\right) + 1\right) K_t^{Z,X}(d\xi)\right)\Bigg] < \infty.$$

HJM framework for the riskfree bond prices

Definition

A bond price model is a quintuple $(B, f_0, \widetilde{\alpha}, \widetilde{\sigma}, X)$, where

- the bank account B satisfies $B_t = e^{\int_0^t r_s ds}$, with short rate r,
- X is a d-dimensional Itô-semimartingale,
- $f_0\colon \mathbb{R}_+ \to \mathbb{R}$ is measurable and $\int_0^T |f_0(t)|dt < \infty$ for any $T \in \mathbb{R}_+$,
- $(\omega, t, T) \mapsto \widetilde{\alpha}_t(T)(\omega)$ and $(\omega, t, T) \mapsto \widetilde{\sigma}_t(T)(\omega)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+)$ measurable $\mathbb{R}$ and $\mathbb{R}^d$-valued processes and satisfy certain integrability conditions,
- the forward rate process $f_t(T)$ is defined by
  $f_t(T) = f_0(T) + \int_0^t \widetilde{\alpha}_s(T)ds + \int_0^t \widetilde{\sigma}_s(T)dX_s$,
- the bond prices $\{(B(t,T))_{t\in[0,T]}, T \geq 0\}$ satisfy $B(t,T) = e^{-\int_t^T f_t(s)ds}$., in particular $B(t,t) = 1$.

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- X is a d-dimensional Itô-semimartingale,
- f_0: $\mathbb{R}_+ \to \mathbb{R}$ is measurable and $\int_0^T |f_0(t)|dt < \infty$ for any $T \in \mathbb{R}_+$,
- $(\omega, t, T) \mapsto \widetilde{\alpha}_t(T)(\omega)$ and $(\omega, t, T) \mapsto \widetilde{\sigma}_t(T)(\omega)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+)$ measurable $\mathbb{R}$ and $\mathbb{R}^d$-valued processes and satisfy certain integrability conditions,
- the forward rate process $f_t(T)$ is defined by $f_t(T) = f_0(T) + \int_0^t \widetilde{\alpha}_s(T)ds + \int_0^t \widetilde{\sigma}_s(T)dX_s$,
- the bond prices $\{(B(t,T))_{t\in[0,T]}, T \geq 0\}$ satisfy $B(t,T) = e^{-\int_t^T f_t(s)ds}$., in particular $B(t,t) = 1$.

An bond price model is called risk neutral if the discounted bond prices $\left\{\left(\frac{B(t,T)}{B_t}\right)_{t\in[0,T]}\right\}$ are martingales.

HJM framework for the riskfree bond prices

Proposition (cf. J. Teichmann's presentation on the CNKK-approach)

A bond price model can be identified with an HJM-type model $(Z, \eta_0, \alpha, \sigma, X)$ *for the family of discounted bond prices* $\left\{\left(\frac{B(t,T)}{B_t}\right)_{t\in[0,T]}\right\}$ *by setting* $\eta_0 = -f_0$, $\alpha = -\widetilde{\alpha}$, $\sigma = -\widetilde{\sigma}$ *(thus* $\eta_t(t) = -f_t(T)$*) and* $Z_t = -\log B_t = -\int_0^t r_s ds$.
Moreover, the following assertions are equivalent:

- *The bond price model is risk neutral, i.e.,* $\left(\frac{B(t,T)}{B_t}\right)_{t\in[0,T]}$ *are martingales for all* $T \geq 0$.
- $\mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right] = e^{Z_t + \int_t^T \eta_t(s)ds} \Leftrightarrow \mathbb{E}\left[\frac{B_t}{B_T}|\mathcal{F}_t\right] = e^{-\int_t^T f_t(s)ds}$.
- *The following conditions hold:*
  - *martingale property of*
    $$\left(\exp\left(\int_0^t \left(-\int_s^T \widetilde{\sigma}_s(u)du\right) dX_s - \int_0^t \Psi_s^X\left(-\int_s^T \widetilde{\sigma}_s(u)du\right) ds\right)\right)_t,$$
  - *Consistency condition:* $\Psi_t^Z(1) = -r_t = -f_t(t)$,
  - *HJM drift condition:* $\int_t^T \widetilde{\alpha}_t(s)ds = \Psi_t^X(-\int_t^T \widetilde{\sigma}_t(s)ds)$.

Remark

- The introduction of a bank account is actually not necessary.
- One could also take the terminal bond $B(t, T^*)$ as numeraire. Then $\left(\frac{B(t,T)}{B(t,T^*)}\right)_{t\in[0,T]}$ should be (local) martingales for all $T \leq T^*$ under the T^*-forward measure.
- Similarly we get an HJM-type model $(Z, \eta_0, \alpha, \sigma, X)$ for the family $\left\{\left(\frac{B(t,T)}{B(t,T^*)}\right)_{t\in[0,T]}, T \leq T^*\right\}$ by setting $\eta_0 = -f_0$, $\alpha = -\widetilde{\alpha}$, $\sigma = -\widetilde{\sigma}$ (thus $\eta_t(t) = -f_t(T)$) and

$$Z_t = -\log(B(t, T^*)) = \int_t^{T^*} f_t(s)ds.$$

- A similar drift and consistency condition assure the local martingale property of $\left(\frac{B(t,T)}{B(t,T^*)}\right)_{t\in[0,T]}$ under the T^*-forward measure.

Modeling the term structure of spreads

- $\mathcal{D} = \{\delta_1, \delta_2, \ldots, \delta_m\}$: family of tenors for some $m \in \mathbb{N}$ with $\delta_1 < \delta_2 < \ldots < \delta_m$

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- Aim: Model the term structure of multiplicative spreads between riskfree and risky forward prices $T \mapsto S^{\delta}(t, T)$ given by

$$S^{\delta_i}(t, T) = \frac{1 + \delta_i L_t(T, T + \delta_i)}{1 + \delta_i L_t^D(T, T + \delta_i)}$$

for all $\delta_i \in \{\delta_1, \ldots, \delta_m\}$.

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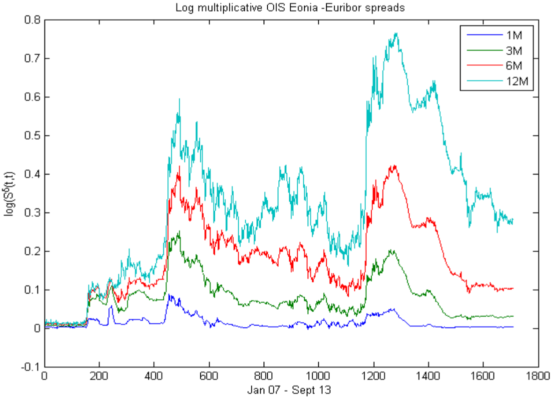
- HJM type models where

$$S^{\delta_i}(t,T) = e^{Z_t^{\delta_i} + \int_t^T \eta_t^i(s)ds}$$

  are particularly suitable because we can model the observed log spot spreads $Z_t^{\delta_i} = \log(S^{\delta_i}(t,t))$ and the forward spread rates $\eta_t^i(T) = \partial_T(\log(S^{\delta_i}(t,T)))$ separately.

OIS Eonia-Euribor spread

- Logarithm of the multiplicative spread $S^{\delta}(t,t)$ from Jan. 2007 to September 2013 for $\delta = 1/12, 3/12, 6/12, 1$:



Modeling the log spot spreads

- Due to a high correlation between the different spreads, principal component analysis (PCA) suggests to model the different log spot spreads by a common lower dimensional process Y taking values in $\mathbb{R}^n$ with $n < m$ (typically $n = 1$ or 2 is sufficient) such that

$$Z_t^{\delta_i} = u_i^\top Y_t,$$

where $u_1, \ldots, u_m$ are some vector in $\mathbb{R}^n$ obtained from PCA.

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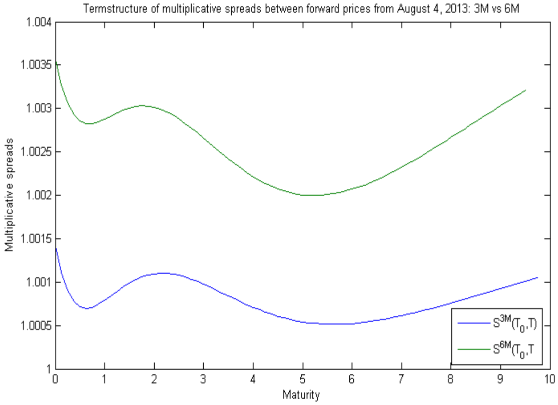
- Ordered spot spreads $1 \leq S^{\delta_1}(t,t) \leq \cdots \leq S^{\delta_m}(t,t)$ can be obtained by taking a process Y which takes values is some cone $C \subset \mathbb{R}^n$ and $u_i \in C^*$ such that $0 < u_1 \prec u_2 \prec \cdots \prec u_m$.

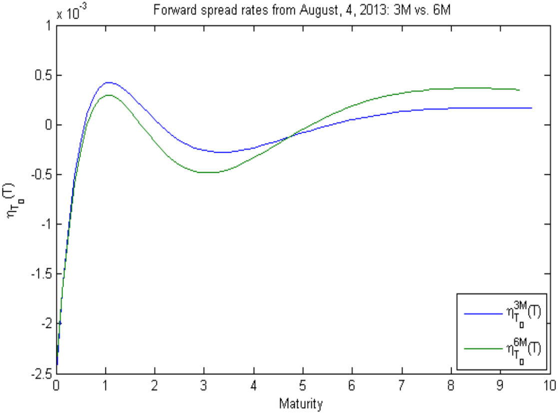
Term structure of multiplicative spreads

- Term structure of multiplicative spreads $S^{\delta}(T_0, T)$ for $\delta = 3/12, 6/12$ at $T_0 = 4.8.2013$



Forward spread rates η

- Forward spread rates $T \mapsto \eta_{T_0}(T)$ for $\delta = 3/12, 6/12$ at $T_0 = 4.8.2013$



Modeling the term structure of $T \mapsto S^{\delta_i}(t, T)$:

- (B_t): bank account
- $B(t, T)$: riskfree bond prices
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Lemma

For every $\delta \in \mathcal{D}$ and $T > 0$, $(S^{\delta}(t, T))_{t \in [0,T]}$ is a $\mathbb{Q}^T$-martingale, where $\mathbb{Q}^T$ denotes the T-forward measure whose density process is given by $\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t B(0,T)}$.

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- In order to model $\{(S^{\delta_i}(t,T))_t, T \geq 0, \delta_i \in \mathcal{D}\}$, the following conditions should thus be satisfied:
  - $(S^{\delta_i}(t,T))_{t\in[0,T]}$ are $\mathbb{Q}^T$-martingales,
  - $S^{\delta_i}(t,T) \geq 1$ for all $t \leq T$ and $T > 0$,
  - $S^{\delta_1}(t,T) \leq \cdots \leq S^{\delta_m}(t,T)$ for all $t \leq T$ and $T > 0$.

Modeling the term structure of $T \mapsto S^{\delta_i}(t,T)$:

- Since $S^{\delta_i}(t,T) = e^{Z_t^{\delta_i} + \int_t^T \eta_t^i(s)ds}$ the $\mathbb{Q}^T$-martingale property implies the conditional expectation hypothesis under $\mathbb{Q}^T$

$$S^{\delta_i}(t,T) = \mathbb{E}_{\mathbb{Q}^T}\left[e^{Z_T^{\delta_i}} \middle| \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}^T}\left[e^{u_i^\top Y_T} \middle| \mathcal{F}_t\right] = e^{u_i^\top Y_t + \int_t^T \eta_t^i(s)ds}.$$

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- We automatically have $1 \leq S^{\delta_1}(t,T) \leq \cdots \leq S^{\delta_m}(t,T)$ for every t and $T \geq t$ if the process Y takes values is some cone $C \subset \mathbb{R}^n$ and $u_i \in C^*$ such that $0 < u_1 \prec u_2 \prec \cdots \prec u_m$, since

$$S^{\delta_i}(t,T) = \mathbb{E}_{\mathbb{Q}^T}\left[e^{u_i^\top Y}\middle| \mathcal{F}_t\right] \leq \mathbb{E}_{\mathbb{Q}^T}\left[e^{u_j^\top Y}\middle| \mathcal{F}_t\right] = S^{\delta_j}(t,T).$$

HJM-type multi-curve models

Definition

- Let the number of different tenors be $m = |\mathcal{D}|$. We call a model consisting of
  - an $\mathbb{R}^{d+n+1}$-valued semimartingale (X, Y, B),
  - vectors $u_1, \dots, u_m$ in $\mathbb{R}^n$,
  - functions f_0, $\eta_0^1, \dots, \eta_0^m$,
  - processes $\widetilde{\alpha}, \alpha^1, \dots, \alpha^m$ and $\widetilde{\sigma}, \sigma^1, \dots, \sigma^m$

  a HJM-type multi-curve model for $\{(B(t,T))_{t \in [0,T]}, T \geq 0\}$ and $\{(S^\delta(t,T))_{t \in [0,T]}, T \geq 0, \delta \in \mathcal{D}\}$ if
  - $(B, f_0, \widetilde{\alpha}, \widetilde{\sigma}, X)$ is a bond price model and
  - for every $i \in \{1, \dots, m\}$, $(u_i^\top Y, \eta_0^i, \alpha^i, \sigma^i, X)$ is a HJM-type models for $\{(S^{\delta_i}(t,T)), T \geq 0\}$.

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  - for every $i \in \{1, \dots, m\}$, $(u_i^\top Y, \eta_0^i, \alpha^i, \sigma^i, X)$ is a HJM-type models for $\{(S^{\delta_i}(t,T)), T \geq 0\}$.

- An HJM-type multi-curve model is called risk neutral if
  - for all $T > 0$, $(\frac{B(t,T)}{B_t})_t$ is a martingale and
  - for all $i \in \{1, \dots, m\}$ and for all $T \geq 0$, $(S^{\delta_i}(t,T))_t$ is a $\mathbb{Q}^T$-martingale.

Multi-curve models - drift and consistency condition

Theorem

For a multi-curve model the following conditions are equivalent:

- *The multi-curve model is risk neutral.*
- *The following conditional expectation hypotheses hold:*

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{B_t}{B_T}|\mathcal{F}_t\right] = e^{-\int_t^T f_t(s)ds}$$

$$\mathbb{E}_{\mathbb{Q}^T}\left[e^{u_i^\top Y_T}|\mathcal{F}_t\right] = e^{u_i^\top Y_t + \int_t^T \eta_t^i(s)ds}, \qquad \textit{for all } i \in \{1,\dots,m\}.$$

Multi-curve models - drift and consistency condition

Theorem (continued)

- *The following conditions are satisfied:*
  - *martingale property (under $\mathbb{Q}$) of*
    - $\left(\exp\left(\int_0^t\left(-\int_s^T \widetilde{\sigma}_s(u)du\right)dX_s - \int_0^t \Psi_s^X\left(-\int_s^T \widetilde{\sigma}_s(u)du\right)ds\right)\right)_t$ *and*
    - $\left(\exp\left(u_i^\top Y_t + \int_0^t\left(\int_s^T(\sigma_s^i(u)-\widetilde{\sigma}_s(u))du\right)dX_s + \right.\right.$ $\left.\left. - \int_0^t \Psi_s^{Y,X}\left(u_i, \int_s^T(\sigma_s^i(u)-\widetilde{\sigma}_s(u))du\right)ds\right)\right)_t,$
  - *Consistency conditions:* $r_t = f_t(t)$ *and* $\Psi_t^Y(u_i) = \eta_{t-}^i(t)$.
  - *HJM drift conditions:*
    - $$\int_t^T \widetilde{\alpha}_t(s)ds = \Psi_t^X\left(-\int_t^T \widetilde{\sigma}_t(s)ds\right)$$
    - $$\int_t^T \alpha_t^i(s)ds = \Psi_t^Y(u_i) - \Psi^{Y,X}\left(u_i, \int_t^T(\sigma_t^i(s)ds - \widetilde{\sigma}_t(s))ds\right) + \\ + \Psi_t^X\left(-\int_t^T \widetilde{\sigma}_t(s)ds\right)$$

Construction of multi-curve models

- Aim: Specify a risk neutral multi-curve model via $(f_0, \eta_0^i, \widetilde{\sigma}, \sigma^i, X, Y)$ such that

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- The second aim can be achieved by taking a process Y which takes values is some cone $C \subset \mathbb{R}^n$ and $u_i \in C^*$ such that $0 < u_1 \prec u_2 \prec \cdots \prec u_m$,

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- The second aim can be achieved by taking a process Y which takes values is some cone $C \subset \mathbb{R}^n$ and $u_i \in C^*$ such that $0 < u_1 \prec u_2 \prec \cdots \prec u_m$,
- The more difficult part is to satisfy the consistency condition

$$\Psi_t^Y(u_i) = \eta_{t-}^i(t).$$

Construction of multi-curve models

- In order to specify the dynamics η^i we need to define the drift α^i as

$$\alpha_t^i(T) = -\partial_T \Psi^{Y,X}\left(u_i, \int_t^T (\sigma_t^i(s)ds - \widetilde{\sigma}_t(s))ds\right) + \partial_T \Psi_t^X\left(-\int_t^T \widetilde{\sigma}_t(s)ds\right).$$

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- For this we can decompose Y into its dependent part $Y^{||}$ relative to X and a locally independent part $Y^{\perp} = Y - Y^{||}$. To define α^i it is sufficient to specify only the dependent part $Y^{||}$ because

$$\Psi^{Y,X} = \Psi^{Y^{||},X} + \Psi^{Y^{\perp},0}.$$

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- Therefore we can specify $(\eta^i_0, \widetilde{\sigma}, \sigma^i, X, Y^{||})$ such that $Y^{||}$ lies in C and $\left(\exp\left(u_i^\top Y^{||}_t + \int_0^t \left(\int_s^T (\sigma^i_s(u) - \widetilde{\sigma}_s(u))du\right) dX_s + \right.\right.$ $\left.\left. - \int_0^t \Psi^{Y^{||},X}_s \left(u_i, \int_s^T (\sigma^i_s(u) - \widetilde{\sigma}_s(u))du\right) ds\right)\right)_t$ is a martingale.

Construction of multi-curve models

- Supposing existence and uniqueness for η^i, we then have to construct $Y^\perp$ with state space C, locally independent of $(Y^{||}, X)$ such that

$$\Psi_t^{Y^\perp}(u_i) = \eta_t^i(t) - \Psi_t^{Y^{||}}(u_i).$$

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  for all i.
- Possible solutions:
  - If $m = n$, $c^{Y^\perp}$ and $K^{Y^\perp}$ could be fixed and the drift chosen accordingly $\Rightarrow$ Problem: $Y^\perp$ should be C-valued.
  - If $m > n$, adjusting only the drift does not work any more.
  - Adjusting the compensator of the jumps allows for highest flexibility, however one has to find a way to guarantee that $Y^\perp \in C$.

Existence of multi-curve models

- It is possible to construct multi-curve models such that all requirements of Condition (iii) (drift and consistency condition and martingale property) are satisfied. Thus the spreads $S^{\delta_i}(t, T)$ are $\mathbb{Q}^T$ martingales.
- Moreover, the process $Y = Y^{||} + Y$ can be specified to take values in C, whence the spreads are ordered.

Definition of an affine Markov process

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Definition (Affine Markov process)

A time-homogeneous Markov process X relative to some filtration $(\mathcal{F}_t)$ and with state space D is called affine if

1. it is stochastically continuous, that is, the transition kernels satisfy $\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly on D for every $t \geq 0$ and $x \in D$, and
2. its Fourier-Laplace transform has exponential-affine dependence on the initial state. This means that there exist functions $\phi : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{C}$ and $\psi : \mathbb{R}_+ \times \mathcal{U} \to V + \mathrm{i}V$ such that for all $x \in D$ and $(t, u) \in \mathbb{R}_+ \times \mathcal{U}$

$$\mathbb{E}_x \left[e^{\langle u, X_t \rangle} \right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\phi(t,u) + \langle \psi(t,u), x \rangle}.$$

Properties of affine processes

Theorem (Keller-Ressel, Teichmann, Schachermayer 2011; C. and Teichmann 2012)

Every affine process X is regular, that is, for every $u \in \mathcal{U}$ the derivatives

$$F(u) := \left.\frac{\partial \phi(t,u)}{\partial t}\right|_{t=0}, \qquad R(u) := \left.\frac{\partial \psi(t,u)}{\partial t}\right|_{t=0}$$

exist and are continuous in u. Moreover, F and R are of Lévy Kinthchine form and ϕ and ψ satisfy the so-called generalized Riccati equations.

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Lemma

Consider an affine process (X, Y) on some mixed state space $D_1 \times D_2$ with scalar product $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ such that the characteristics of Y only depend on X. Then

$$\mathbb{E}\left[e^{\langle u, X_t \rangle_1 + \langle v, Y_t \rangle_2}\right] = e^{\phi(t,u,v) + \langle \psi(t,u,v), x \rangle_1 + \langle v, y \rangle_2}.$$

Affine multi-curve model

Definition

An affine multi-curve model is defined via

- an affine process (X, Y, Z) on some state space $D \subset \mathbb{R}^{d+n+1}$ satisfying certain exponential moment conditions with the property that the characteristics of (Y, Z) only depend on X, in particular $Z_t = -\int_0^t r_s ds = -\int_0^t l + \langle \lambda, X_s \rangle ds$ such that
- the bank account satisfies $B_t = e^{-Z_t} = e^{\int_0^t r_s ds}$,
- the bond prices satisfy

$$B(t,T) = \mathbb{E}\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right] = \mathbb{E}\left[e^{Z_T - Z_t}|\mathcal{F}_t\right] = e^{\phi(T-t,0,0,1)+\langle \psi(T-t,0,0,1), X_t \rangle},$$

- for each i, the spreads $S^{\delta_i}(t,T)$ satisfy

$$S^{\delta_i}(t,T) := \frac{\mathbb{E}\left[e^{Z_T + u_i^\top Y_T}|\mathcal{F}_t\right]}{\mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right]}$$
$$= e^{u_i^\top Y_t + \phi(T-t,0,u_i,1) - \phi(T-t,0,0,1) + \langle \psi(T-t,0,u_i,1) - \psi(T-t,0,0,1), X_t \rangle}$$

Relation to HJM-type multi-curve models

Proposition

Every affine multi-curve model is a risk neutral HJM-type multi-curve model where

- *the driving process is* X,
- *the bank account is given by* $B_t = e^{-Z_t}$
- *the log spot spread is given by* $\log(S^{\delta_i}(t,t)) = u_i^\top Y_t$ *and*
- *the forward rate and forward spread rates are given by*

$$
\begin{aligned}
f_t(T) &= -F(\psi(T-t,0,0,1),0,1) - \langle R(\psi(T-t,0,0,1),0,1), X_t\rangle \\
\eta_t^i(T) &= F(\psi(T-t,0,u_i,1),u_i,1) - F(\psi(T-t,0,0,1),0,1) \\
&\quad + \langle R(\psi(T-t,0,u_i,1),u_i,1) - R(\psi(T-t,0,0,1),0,1), X_t\rangle
\end{aligned}
$$

Pricing of interest rate derivatives

- Pricing of FRA contracts, swaps and basis swaps amounts to compute riskfree bond prices and the following quantity

$$\begin{aligned} B(t,T)S^{\delta_i}(t,T) &= \mathbb{E}_{\mathbb{Q}}[e^{u_i^\top Y_T + Z_T - Z_t} | \mathcal{F}_t] \\ &= e^{\phi(T-t,0,u_i,1)+\langle \psi(T-t,0,u_i,1), X_t\rangle - Z_t}, \end{aligned}$$

which simply means solving the Riccati equations for ϕ and ψ.

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$$\begin{aligned} B(t,T)S^{\delta_i}(t,T) &= \mathbb{E}_{\mathbb{Q}}[e^{u_i^\top Y_T + Z_T - Z_t} \mid \mathcal{F}_t] \\ &= e^{\phi(T-t,0,u_i,1) + \langle \psi(T-t,0,u_i,1), X_t \rangle - Z_t}, \end{aligned}$$

  which simply means solving the Riccati equations for ϕ and ψ.
- Pricing of caplets can be achieved via Fourier methods as for pricing put options in affine models.

Relation to other models

- Lognormal LIBOR market models
  - Similarly as in the original BGM article, we can obtain a lognormal LIBOR market model for $L_t(T, T+\delta)$ within the above framework.
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- Multi-curve HJM models
  - The HJM multiple-curve models recently proposed by Crepey et al. and Moreni and Pallavicini can also be recovered within our framework.

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- Work in progress, Outlook
  - Statistical analysis of the dependence and correlation structure between the different curves and spreads
  - Calibration

- Thank you for your attention!